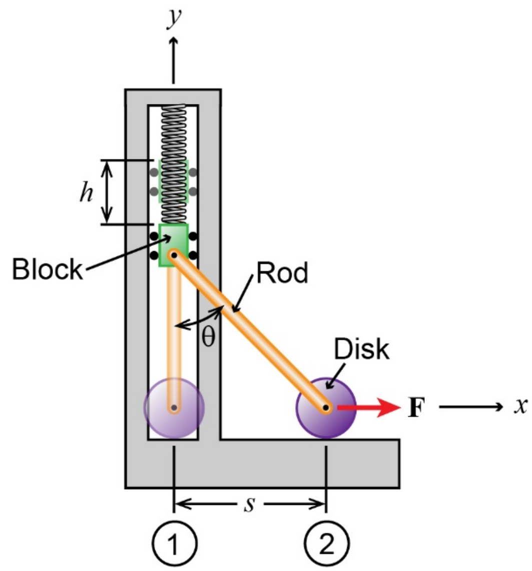
Solved Problem 8.5-7

The system shown consists of a block of mass m_B, a slender rod of length L and mass m_L, and a disk of radius r and mass m_R. The block is attached to a linear spring of stiffness k and is confined to move within a vertical shaft with negligible friction. The disk or rolls, without slip, on a horizontal floor. The rod connects the block with the center of the disk. The system starts from rest, the spring is un-stretched when the rod is vertical, and a constant horizontal force **F** is applied at the disk's center at state 1. Determine the speed of the disk's center (v_D) as a function of θ and F and the following physical parameters of the system (m_B, m_L, m_D, L, k).



Given: Block parameters: m_B, no friction
Slender rod parameters: m_L, L
Disk parameters: m_D, r
Spring stiffness = k

Find: v_D as a function of θ, F, m_B, m_L, m_D, L, k

Solution

Getting familiar with the system

Since the problem statement specifically asks for a velocity given an applied force, this gives us a good indication that the energy method may be our best bet for solving this problem. Given that there is an externally applied force, this is not a conservative system.

Work-energy balance

Since the system starts from rest, the kinetic energy at state 1 is zero. We will also take the gravitational potential energy at state 1 to be zero. The elastic potential energy at state 1 is also zero since the spring is un-stretched at this point.

$$\cancel{T_1} + \cancel{V_1} + U_{1-2} = T_2 + V_2$$

$$Fs = \frac{1}{2}m_B v_{B,2}^2 + \frac{1}{2}m_R v_{GR,2}^2 + \frac{1}{2}I_{GR}\omega_{R,2}^2 + \frac{1}{2}m_D v_{GD,2}^2 + \frac{1}{2}I_{GD}\omega_{D,2}^2 - m_B gh - m_R gh_{GR} + \frac{1}{2}kh^2$$

KE of the block | KE of the rod | KE of the disk | Potential en

The above equation is the overall work-energy balance equation of the system. Let's break it up and look at each individual piece and relate each of the variables to v_D, θ, F, m_B, m_L, m_D, L, and k.

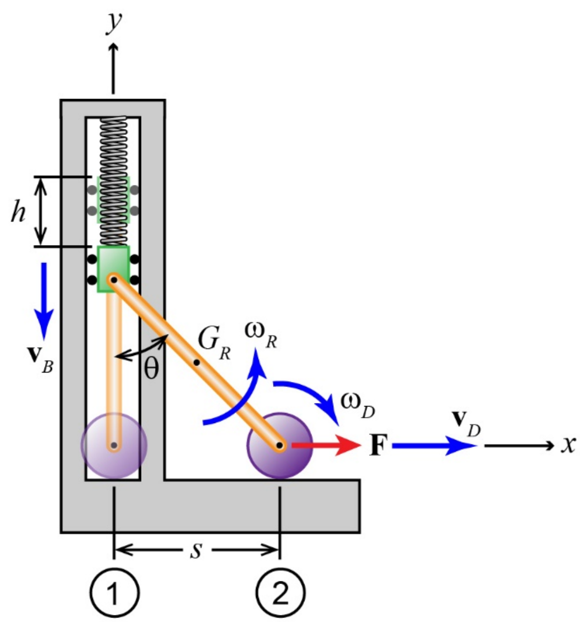
Work

The work done by force $\mathbf{F}$ is simply the force multiplied by distance. However, we need relate that distance to the angle θ.

$$U_{1-2} = Fs = FL\sin\theta$$

Kinetic energy of the disk

The disk both translates and rotates, therefore, the kinetic energy equation must capture both of those energies. Remember that the disk's velocity is measured at the mass moment of inertia is in reference to the disk's center of gravity/mass.



$$T_{disk} = \frac{1}{2}m_D v_D^2 + \frac{1}{2}I_{GD}\omega_D^2$$

The mass moment of inertia of a slender rod relative to its center of mass may be obtained from Appendix A.

$$I_{GD} = \frac{1}{2}m_D r^2$$

The problem statement indicates that the disk rolls without slip, therefore,

$$v_D = \omega_D r$$

Substituting these relationships into the kinetic energy equation we get

$$T_{disk} = \frac{1}{2}m_D v_D^2 + \frac{1}{2}\left(\frac{1}{2}m_D r^2\right)\left(\frac{v_D}{r}\right)^2 = \frac{1}{2}m_D v_D^2 + \frac{1}{4}m_D v_D{}^2 = \frac{3}{4}m_D v_D{}^2$$

Kinetic energy of the block

The block may be treated as a particle because it does not rotate. Therefore, its kinetic energy takes the form

$$T_{block} = \frac{1}{2} m_B v_B^2$$

We will use kinematics to relate v_B to the speed of the disks center v_D.

$$\mathbf{v}_B = \mathbf{v}_D + \boldsymbol{\omega}_R \times \mathbf{r}_{B/D} = v_D\,\mathbf{i} + \omega_R \times L(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$$
$$v_B(-\mathbf{j}) = v_D\,\mathbf{i} + \omega_R L(-\sin\theta\,\mathbf{j} - \cos\theta\,\mathbf{i})$$

Equating the $\mathbf{i}$ and $\mathbf{j}$ components of the equation we get

$$\mathbf{i} : v_D = \omega_R L\cos\theta$$
$$\mathbf{j} : v_B = \omega_R L\sin\theta = v_D\tan\theta$$

Substituting this relationship into the kinetic energy equation we obtain

$$T_{block} = \frac{1}{2} m_B v_D^2 \tan^2\theta$$

Kinetic energy of the rod

The rod both translates and rotates, therefore, the kinetic energy equation must capture both of those energies. Remember that the rod's velocity is measured at the mass moment of inertia is in reference to the rod's center of gravity/mass.

$$T_{rod} = \frac{1}{2} m_R v_{GR}^2 + \frac{1}{2} I_{GR}\omega_R^2$$

The mass moment of inertia of a slender rod relative to its center of mass may be obtained from Appendix A.

$$I_{GR} = \frac{1}{12} m_R L^2$$

The velocity of the rod's center of mass may be obtained by applying kinematics.

$$\mathbf{v}_B = \mathbf{v}_{GR} + \boldsymbol{\omega}_R \times \mathbf{r}_{B/GR}$$
$$v_B(-\mathbf{j}) = \mathbf{v}_{GR} + \omega_R \times \frac{L}{2}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = \mathbf{v}_{GR} + \omega_R\frac{L}{2}(-\sin\theta\,\mathbf{j} - \cos\theta\,\mathbf{i})$$
$$\mathbf{v}_{GR} = (\omega_R\frac{L}{2}\sin\theta - v_B)\,\mathbf{j} + \omega_R\frac{L}{2}\cos\theta\,\mathbf{i}$$

Using the relationship between v_B and ω_R derived earlier, we get

$$\mathbf{v}_{GR} = (\omega_R \frac{L}{2}\sin\theta - \omega_R L \sin\theta)\,\mathbf{j} - \omega_R \frac{L}{2}\cos\theta\,\mathbf{i} = \omega_R \frac{L}{2}(\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j})$$

Therefore, the magnitude of $\mathbf{v}_{GR}$ is given below remembering that $\cos^2\theta + \sin^2\theta = 1$ and $\omega_R = v_D / L\cos\theta$.

$$v_{GR} = \frac{v_D}{2\cos\theta}$$

Plugging the above relationships into the kinetic energy equation we get

$$T_{rod} = \frac{1}{2}m_R v_{GR,2}^2 + \frac{1}{2}I_{GR}\omega_{R,2}^2 = \frac{1}{2}m_R\left(\frac{v_D}{2\cos\theta}\right)^2 + \frac{1}{2}\frac{1}{12}m_R L^2\left(\frac{v_D}{L\cos\theta}\right)^2$$
$$= \frac{1}{8}m_R\left(\frac{v_D}{\cos\theta}\right)^2 + \frac{1}{24}m_R\left(\frac{v_D}{\cos\theta}\right)^2 = \frac{1}{6}m_R\left(\frac{v_D}{\cos\theta}\right)^2$$

Potential energy

The potential energy at state 2 or the change in potential energy from state 1 to state 2 consists of the drop in height of the block, the drop in height of the mass center of the rod and the stretching of the spring by the amount that the block drops.

$$V_2 = -m_B g h - m_R g h_{GR} + \frac{1}{2}kh^2 = -m_B g L(1-\cos\theta) - m_R g\frac{L}{2}(1-\cos\theta) + \frac{1}{2}kL^2(1-\cos\theta)^2$$
$$= -\left(m_B + \frac{m_R}{2}\right)gL(1-\cos\theta) + \frac{1}{2}kL^2(1-\cos\theta)^2$$

Substituting all the energies back into the work-energy balance equation and solving for v_D we get

$$Fs = \frac{1}{2}m_B v_{B,2}^2 + \frac{1}{2}m_R v_{GR,2}^2 + \frac{1}{2}I_{GR}\omega_{R,2}^2 + \frac{1}{2}m_D v_{GD,2}^2 + \frac{1}{2}I_{GD}\omega_{D,2}^2 - m_B g h - m_R g h_{GR} + \frac{1}{2}kh^2$$

$$v_D^2 = \frac{L\cos^2\theta\big((2m_B + m_R)g(1-\cos\theta) - kL(1-\cos\theta)^2 + 2F\sin\theta\big)}{3\big(3m_B\sin^2\theta + m_R + 9m_D\cos^2\theta\big)}$$